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TRANSPORT PROCESSES IN FRACTALS— IV. NONLINEAR FLOW PROBLEMS IN FRACTAL CAPILLARY NETWORKS

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Abstract—The construction of a fractal capillary network is briefly recalled. The nonlinear transfer function of the basic graph is calculated. Then, a general method is given to obtain the transfer function of a fractal; it depends upon its algebraic description, the nonlinear character of the fluid and the basic graph. The threshold for a Bingham fluid to flow is also determined. Power-law fluids and Bingham fluids are then studied on a Sierpinski gasket.

1. INTRODUCTION

In the present paper, we shall address the problem of the flow of non-Newtonian fluids in fractal capillary networks. Hence, this series about fractals (Adler 1985, a, b, c, hereafter referred to as I, II and III) parallels our previous series about spatially periodic capillary networks (Adler & Brenner 1984 a, b, c, hereafter referred to as IV, V, VI). Again, we shall demonstrate that the methods developed in II and IV for linear flow problems can be successfully applied, after some substantial modifications, to nonlinear situations (see IV for such an extension to spatially periodic networks). As in VI, two quite different cases have been selected for study here: the first involves non-Newtonian (or, possibly, inertial) effects; the second, the derivation of threshold criteria for establishing the onset (or cessation) of flow through the network as a consequence of the existence of a "microscopic" nonzero yield stress, below which no flow occurs within an individual capillary.

To the best of our knowledge, there is no previous contribution about nonlinear phenomena in fractals. All the references cited in I, II and III are exclusively concerned with linear phenomena (possibly time dependent). However, the chaotic behaviour of nonlinear dynamic systems is now often related to fractal mathematical structures (cf. for instance Hu, 1982).

The organization of this paper is as follows. In order to avoid the search of the relevant informations in I–VI, it is self-contained within reasonable limits. The physical background, the major definitions and the methods are given in full, while most of the technical details are omitted.

In section 2, the construction and description of a fractal capillary network is recalled (cf. II) with the basic notations. Sections 3 and 4 are concerned with the flow of non-Newtonian fluids in a network. First, the nonlinear transfer functions on the basic graph are derived with the method used in III; some of their general properties are given. Then, in section 4, the transfer functions of the various generations of the fractal are derived. As an example, a power-law fluid is studied on a Sierpinski gasket; a fractal constant is obtained which depends upon the fluid index; this is the first result of this kind to our knowledge.

Finally, section 5 deals with Bingham fluids. Addressed here during the resolution of such problems are the following fundamental questions: What minimal pressure difference is required for the flow to commence? Through which capillaries does it occur? This is addressed first on the basic graph and second on the fractal graph. Finally, the example of the Sierpinski gasket is briefly addressed.

In this paper, some familiarity with graph theory is assumed. Only a few definitions are given, and the reader is referred to IV and to Biggs (1974) for further definitions and properties.

2. GEOMETRICAL DESCRIPTION OF A FRACTAL CAPILLARY NETWORK

The general construction of a fractal capillary network was fully detailed in II, to which the reader is referred. We shall restrict ourselves here to a short presentation of the construction process and to an introduction of the main notations.

The basic unit of a fractal capillary network is a basic graph Γ_0, which is a finite graph. The infinite replication of Γ_0 according to certain laws constitutes the fractal. Γ_0 may be considered as

$$\begin{aligned} &\text{a set } V\Gamma_0 \text{ of } n_0 \text{ vertices } v_i \text{ connected by} \\ &\text{a set } E\Gamma_0 \text{ of } m_0 \text{ edges } e_j. \end{aligned} \qquad [1]$$

A few general definitions may be given here. Two vector spaces called the edge space and the vector space of dimension m_0 and n_0 may be introduced; they correspond to the edge and vertex sets, respectively (cf. IV). Vectors are defined on these spaces in the following way; for instance, a flow rate vector J may be defined on the edge space; its jth component is the flow rate on the edge j. The edge space may be decomposed into two orthogonal subspaces, called the cycle and co-cycle subspaces; vectors of the edge space which belong to these subspaces are called cycle or co-cycle vectors.

Two kinds of vertices are distinguished in Γ_0 (cf. figure 1a):

$$\begin{aligned} &n_e \text{ external vertices } v_i^{(e)} \\ &n_0 - n_e \text{ internal vertices } v_i^{(i)}. \end{aligned} \qquad [2]$$

G_0 is generally assumed to be connected.

Consider a given family of transformations S_α ($\alpha = 1, \ldots, M$) which act on the basic graph Γ_0, in a manner which will be specified later. The construction of the graph Γ_1 requires two steps (cf. figure 1). First, the graph Γ_1' is defined as the juxtaposition of the graphs $S_\alpha\Gamma_0$

$$\Gamma_1' = \overset{M}{\underset{i=1}{U'}} S_\alpha\Gamma_0, \qquad [3]$$

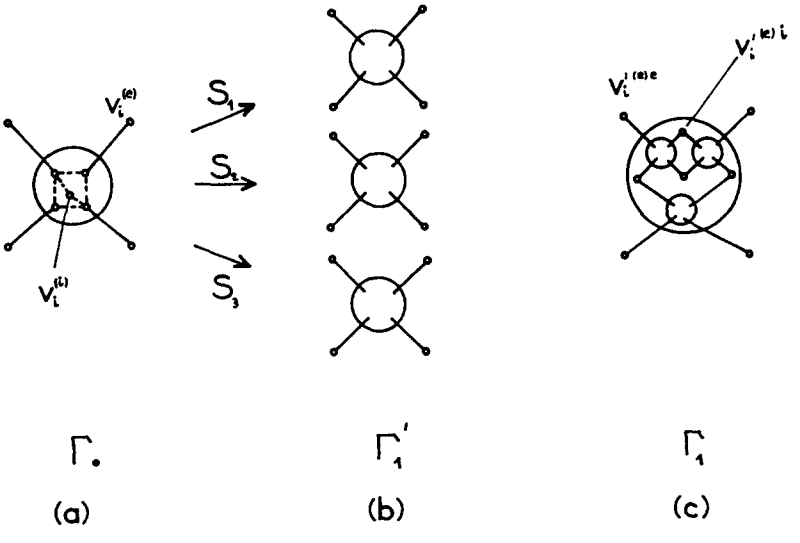

Figure 1. Construction of a fractal. (a) The basic graph Γ_0 with its n_e external vertices $v_i^{(e)}$ and its $n_0 - n_e$ internal vertices $v_i^{(i)}$. (b) The transformations S_α ($\alpha = 1, \ldots, M$) yield the graph Γ_1' made of the juxtaposition of the M transformed graphs $S_\alpha\Gamma_0$. (c) These graphs are then interconnected in a specified way to form the graph Γ_1. Note the distribution between the external vertices $v_i^{(e)i}$ of Γ_1' which become internal vertices of Γ_1, and the external vertices $v_i^{(e)e}$ of Γ_1' which become external vertices of Γ_1.

where U' denotes the juxtaposition without any interconnections between the external vertices of the graphs $S_\alpha\Gamma_0$.

Second, the external vertices of the elementary graphs $S_\alpha\Gamma_0$ are connected one to the other one in a specified way

$$\Gamma_1 = \overset{M}{\underset{i=1}{U}} S_\alpha\Gamma_0. \qquad [4]$$

Let us assume that the interconnections are defined in such a way as to leave n_e external vertices to the graph Γ_1. Hence, the process can be indefinitely continued; Γ_N may be expressed as:

$$\Gamma_N = \overset{M}{\underset{\alpha=1}{U}} S_\alpha\Gamma_{N-1}. \qquad [5]$$

The interconnections between the $(M-1)n_e$ external vertices $v_i'^{(e)i}$ of Γ_1' which become internal vertices of Γ_1 was symbolized by the $(M-1)n_e \times (M-1)n_e$ matrix ζ defined as (cf. II):

$$\begin{aligned} \zeta_{ij} &= 1 \quad \text{when the vertices } v'^{(e)i} \text{ and } v_j'^{(e)i} \text{ are superposed} \\ &= 0 \quad \text{otherwise.} \end{aligned} \qquad [6]$$

The transformations S_α can belong to a very large class. Usually, it consists of the multiplication of all the lengths involved in a given geometry by a factor λ (cf. II). In the examples under consideration here, S_α is the identity.

3. THE NONLINEAR TRANSFER FUNCTIONS OF THE BASIC GRAPH

This section is devoted to the elementary analysis of flow of nonlinear fluids on the basic graph Γ_0. The nonlinear fluids which are considered here belong to the class of the so-called generalized Newtonian fluids; the study of Bingham fluids is postponed to section 5. In the first paragraph, the general properties of the transfer functions are given. These functions are derived in the second paragraph, which may be skipped.

3.1. *General*

Consider the piece of porous material symbolized by the graph Γ_0. As in II, we are only interested by the relation between the flow rates going in and out of the material and the pressures imposed at the external vertices.

To each external vertex of Γ_0 can be associated a pressure and a flow rate, with the arbitrary convention that the flow rate is positive when it goes out of Γ_0 (see figure 2a). Hence, the n_e pressures and the n_e flow rates can be represented by the vectors $\mathbf{P}_0^{(e)}$ and $\mathbf{J}_0^{(e)}$, respectively, which are defined on the space of the external vertices $V\Gamma_0^{(e)}$ (i.e. the ith component of $\mathbf{P}_0^{(e)}$ is the pressure at the ith external vertex); the relation between these two vectors may be expressed as

$$\mathbf{J}_0^{(e)} = \mathbf{A}_0(\mathbf{P}_0^{(e)}), \qquad [7]$$

where $\mathbf{A}_0$ is a nonlinear vectorial function, with n_e components. It is called the nonlinear transfer matrix of the graph Γ_0.

$\mathbf{A}_0$ possesses general properties which are similar to the properties of its linear counterpart $\mathbf{A}_0$ which was introduced in II. First, the flow rates do not vary when an arbitrary

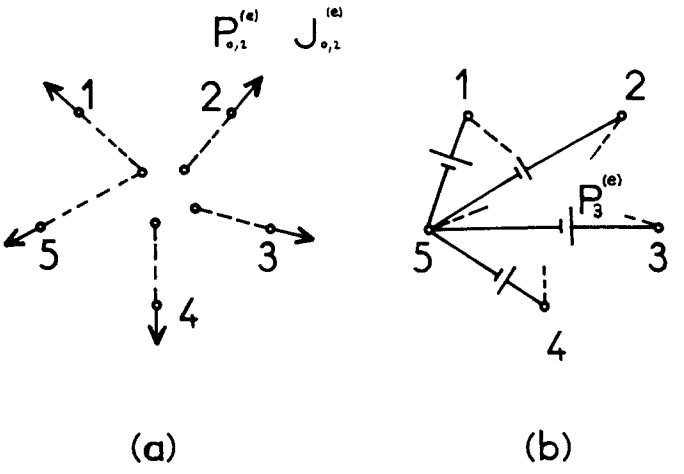

Figure 2. Flow on the basic graph Γ_0. (a) The external vertices 1 to 5 are represented together with the corresponding pressures $P_{0,i}^{(e)}$ and outgoing flow rates $J_{0,i}^{(e)}$ ($i = 1, \ldots, 5$). (b) The graph γ_0 on which the function $\mathbf{a}_0$ is determined. Generators whose strength is equal to the relative pressures are positionned between $n_e - 1$ external vertices and for instance the last one.

constant K is added to the pressures; hence,

$$\mathbf{A}_0(\mathbf{P}_0^{(e)} + K\mathbf{1}) = \mathbf{A}_0(\mathbf{P}_0^{(e)}), \forall \mathbf{P}_0^{(e)}, K, \qquad [8]$$

where $\mathbf{1}$ denotes the vector, $\underbrace{(1 \ldots 1)}_{n_e}^{\dagger}$ on $V\Gamma_0^{(e)}$.

Second, the sum of the outgoing flow rates is equal to zero as a direct consequence of the conservation of the flow rates at each vertex of the graph Γ_0. Thus,

$$\mathbf{1}^r \cdot \mathbf{A}_0(\mathbf{P}_0^{(e)}) = 0, \forall \mathbf{P}_0^{(e)}. \qquad [9]$$

The function $\mathbf{A}_0$ is noninvertible, for exactly the same reasons as its linear equivalent $\mathbf{A}_0$ (cf. II). Again, this unessential undeterminacy can be removed by choosing an arbitrary vertex, say the last one $v_{n_e}^{(e)}$ which is assumed to be at a zero pressure. Moreover, the flow rate $J_{0,n_e}^{(e)}$ is given by:

$$J_{0,n_e}^{(e)} = -\sum_{i=1}^{n_e-1} J_{0,i}^{(e)}. \qquad [10]$$

The remaining unknowns ($J_{0,i}^{(e)}$; $i = 1, \ldots, n_e - 1$) and ($P_{0,i}^{(e)}$; $i = 1, \ldots, n_e - 1$) may be represented by the vectors $\mathbf{j}_0^{(e)}$ and $\mathbf{p}_0^{(e)}$, respectively. They are linked by the relation:

$$\mathbf{j}_0^{(e)} = \mathbf{a}_0(\mathbf{p}_0^{(e)}). \qquad [11]$$

It will be assumed that such a relation can be locally inverted, at least for a limited range of values of $\mathbf{p}_0^{(e)}$ and when a particular solution is possibly chosen:

$$\mathbf{p}_0^{(e)} = \mathbf{a}'^{-1}_0(\mathbf{j}_0^{(e)}), \qquad [12]$$

where the prime reminds the reader that the "local" inverse of $\mathbf{a}_0$ may depend upon the particular range of values of $\mathbf{p}_0^{(e)}$ under consideration and upon the particular solution.

Since the dissipation in each edge of the basic graph Γ_0 is positive, the dissipation in the

whole structure is positive. As a consequence of our conventions, it is readily deduced that:

$$\mathbf{p}_0^{(e)} \cdot \mathbf{a}_0(\mathbf{p}_0^{(e)}) < 0 \text{ if } \mathbf{p}_0^{(e)} \neq 0. \quad [13]$$

Once, $\mathbf{a}_0$ is known, it is not difficult to obtain $\mathbf{A}_0$. Every component of $\mathbf{p}_0^{(e)}$ is replaced by:

$$p_{0,i}^{(e)} = P_{0,i}^{(e)} - P_{0,n_e}^{(e)}. \quad [14]$$

The $n_e - 1$ first components of $\mathbf{J}_0^{(e)}$ are equal to the components of $\mathbf{j}_0^{(e)}$; $J_{0,n_e}^{(e)}$ is obtained through [10].

3.2. *Calculation of the function* $\mathbf{a}_0$

The nonlinear transfer function $\mathbf{a}_0$ can be calculated in a way which is very similar to the one used for the analysis of nonlinear flow problems in spatially periodic capillary networks (cf. VI). All the graph formalism which was introduced in IV, V, VI is not detailed again here.

An other useful tool is the generalized inverse (Lancaster 1969) which was already used in VI. Denote by G any $m \times n$ matrix of rank r and let $\mathbf{f}_1 \ldots \mathbf{f}_r$ be a basis for the image of G. Define $\mathbf{F} = [\mathbf{f}_1 \ldots \mathbf{f}_r]$. Each column of G is a linear combination of the columns of F. Explicitly,

$$\mathbf{G} = \mathbf{F} \cdot \mathbf{R}\dagger, \quad [15]$$

where R is an $n \times r$ matrix.

The generalized inverse of G denoted by $\mathbf{G}^I$ is given explicitly by:

$$\mathbf{G}^I = \mathbf{R} \cdot (\mathbf{R}\dagger \cdot \mathbf{R})^{-1} \cdot (\mathbf{F}^+ \cdot \mathbf{F})^{-1} \cdot \mathbf{F}\dagger. \quad [16]$$

$\mathbf{G}^I$ possesses the usual properties of an inverse (cf. Lancaster 1969 and VI). Morevoer, it may be employed to solve the standard linear equation,

$$\mathbf{G} \cdot \mathbf{x} = \mathbf{b}, \quad [17]$$

where b is a given vector. A solution is easily shown to be:

$$\mathbf{x}_0 = \mathbf{G}_I \cdot \mathbf{b}, \quad [18]$$

where b belongs to the image of G. Of course, any vector belonging to the kernel of G can be added to $\mathbf{x}_0$.

In order to be specific, we shall restrict ourselves to power-law fluids in a first step. These are inelastic fluids whose rheological properties are characterized by a single, shear rate-dependent scalar viscosity function (Astarita & Marucci 1974). The pressure drop along a given edge j is concisely summarized by (cf. VI):

$$\Delta p(j) = -K \cdot s(j) \cdot |J(j)|^{n-1} J(j), \quad [19]$$

where $S(j)$ is the "conductance" of the edge j; it depends, *inter alia*, upon the geometrical characteristics of j. K is the consistency of the fluid. n is the power law index. Note that it is necessary to keep track of the algebraic sign of $J(j)$ in applying [19]. In the future, it will prove convenient to denote by $\mathbf{J}^n$ the vector of the edge space $E\Gamma_0$ whose component is:

$$|J(j)|^{n-1} J(j)$$

on the edge j.

Of course, [19] assumes that the pressure drops corresponding to the junctions between the capillaries and to the entry regions in the capillaries can be neglected; this is certainly so when the length of the capillaries is much larger than their diameter and when there is no singularity associated to the junctions.

With these preliminaries, $\mathbf{a}_0(\mathbf{p}_0^{(e)})$ can be calculated on a modification γ_0 of the basic graph Γ_0 (Bollobas 1979; II), which is illustrated in figure 2.b. Additional edges are added between the first $n_e - 1$ external vertices of Γ_0 and the last one. Generators, whose strength is equal to the relative pressures $\mathbf{p}_0^{(e)}$ are supposed to be located along these edges.

This new graph, denoted by γ_0, possesses n_0 vertices and $\overline{m} = m + n_e - 1$ edges (since the basic graph Γ_0 had n_0 vertices and m edges).

The following vectors may be defined on the edge space $E\gamma_0$ of γ_0. $\bar{\mathbf{j}}_0$ is the flow rate vector whose jth component represents the flow rate $j_0(j)$ along the jth edge of γ_0. $\bar{\mathbf{p}}_0$ is the pressure difference vector, whose elements consist of the pressure differences existing between vertex pairs situated at either end of edge j.

It is assumed that the last $n_e - 1$ edges of γ_0 are the additional edges. Hence, the last $n_e - 1$ components of $\bar{\mathbf{j}}_0$ and $\bar{\mathbf{p}}_0$ are equal to $\mathbf{j}_0^{(e)}$ and $\mathbf{p}_0^{(e)}$, respectively. $\bar{\mathbf{j}}_0$ and $\bar{\mathbf{p}}_0$ may be decomposed into two vectors

$$\bar{\mathbf{j}}_0 = \begin{pmatrix} \mathbf{j}_{\text{int}} \\ \mathbf{j}_0^{(e)} \end{pmatrix}, \quad \bar{\mathbf{p}}_0 = \begin{pmatrix} \mathbf{p}_{\text{int}} \\ \mathbf{p}_0^{(e)} \end{pmatrix}, \tag{20}$$

where $\mathbf{j}_{\text{int}}$ and $\mathbf{p}_{\text{int}}$ are the flow rate and pressure difference vectors on the m first edges of γ_0.

The diagonal $\overline{m} \times \overline{m}$-conductance matrix $\mathbf{M}$ is defined by its non-zero diagonal elements:

$$\begin{aligned} M(j_1, j_2) &= s(j_1)\delta(j_1, j_2) \qquad j_1 \leq m \\ M(j_1, j_2) &= 0 \qquad j_1 > m \text{ or } j_2 > m. \end{aligned} \tag{21}$$

Hence, the conductance of the additional edges is equal to zero.

The pressure generator vector $\mathbf{g}$ is a vector defined on the edge space of γ_0 as:

$$\begin{aligned} g(j) &= 0 && j \leq m \\ &= p_0^{(e)}{}_{,j-m} && \overline{m} \geq j > m \end{aligned} \tag{22}$$

All these notations may be used to express in a compact way the nonlinear Ohm's law on the graph γ_0,

$$\bar{\mathbf{p}}_0 = K\mathbf{M} \cdot \bar{\mathbf{j}}_0^n + \mathbf{g}. \tag{23}$$

The total pressure difference is equal to zero along any cycle of the graph. This property yields a second set of $\overline{m} - n_0 + 1$ equations

$$\mathbf{C}\dagger \cdot \bar{\mathbf{p}}_0 = 0, \tag{24}$$

where $\mathbf{C}$ is the $\overline{m} \times (\overline{m} - n_0 + 1)$ cycle matrix of the graph γ_0 (cf. IV); its columns are independent cycles of the graph γ_0.

Finally, the continuity equation is written for each vertex of γ_0. This yields a third set of $(n_0 - 1)$ equations:

$$\mathbf{K}\dagger \cdot \bar{\mathbf{j}}_0 = 0, \tag{25}$$

where $\mathbf{K}$ is the $\overline{m} \times (n_0 - 1)$ co-cycle matrix of the graph γ_0 (cf. IV and VI); its columns are independent cocycles of the graph γ_0.

Equations [23], [24] and [25] constitute the three basic equations of the problem. They will be solved by a method very similar to the one used in VI. The pressure difference vector $\bar{\mathbf{p}}_0$ may be eliminated in [23] by multiplying this equation by C† to obtain

$$K \cdot \mathbf{C}\dagger \cdot \mathbf{M} \cdot \bar{\mathbf{j}}_0^n + \mathbf{C}\dagger \cdot \mathbf{g} = 0. \qquad [26]$$

Since $\mathbf{C}\dagger \cdot \mathbf{g}$ obviously belongs to the image of C†, equation [26] can be solved by the use of the generalized inverse. Moreover, a simplification occurs in the present case, since the rank of C† is equal to its number of rows. Hence, it is always possible to choose F and G defined by [15] as

$$\mathbf{F} = \mathbf{I}, \mathbf{R}\dagger = \mathbf{C}\dagger \qquad [27]$$

from which the generalized inverse of C† is found to be:

$$(\mathbf{C}\dagger)^I = \mathbf{C} \cdot (\mathbf{C}\dagger \cdot \mathbf{C})^{-1} \qquad [28]$$

Equation [26] may be equivalently written as:

$$\mathbf{M} \cdot \bar{\mathbf{j}}_0^n = -K^{-1} \cdot (\mathbf{Q} \cdot \mathbf{q} + \mathcal{N}) \qquad [29a]$$

where Q is given by:

$$\mathbf{Q} = \mathbf{C} \cdot (\mathbf{C}\dagger \cdot \mathbf{C})^{-1} \cdot \mathbf{C}\dagger \qquad [29b]$$

$\mathcal{N}$ is an arbitrary vector belonging to the kernel of C†. In other words, $\mathcal{N}$ is a co-cycle vector. As such, it possesses only $n_0 - 1$ independent components and may be expressed as:

$$\mathcal{N} = \mathbf{K} \cdot \mathbf{N} \qquad [30]$$

where N is a vector of the co-cycle space with $n_0 - 1$ components.

Equation [26] can now be split into two parts. Its last $n_e - 1$ components yield $n_e - 1$ relations between N and g:

$$(\mathbf{Q} \cdot \mathbf{g} + \mathbf{K} \cdot \mathbf{N})_{,j} = 0, \qquad \overline{m} - n_e + 1 \leq j \leq \overline{m} \qquad [31]$$

The first $\overline{m} - n_e + 1$ components of equation [26] provide the flow rates inside the edges of Γ_0; it can be written as:

$$\mathbf{j}_{\text{int}} = -K^{-1}\mathbf{M}_{\text{int}}^{-1/n} \cdot (\mathbf{Q} \cdot \mathbf{g} + \mathbf{K} \cdot \mathbf{N})^{1/n}, \qquad [32]$$

where $\mathbf{a}^{1/n}$ means a vector formed from the components of a raised to the power of $1/n$, with the prior understanding that one is to keep track of the algebraic sign. $M_{\text{int}}^{-1/n}$ is the $(\overline{m} - n_e + 1) \times (\overline{m} - n_e + 1)$ diagonal matrix defined as:

$$M_{\text{int}}^{-1/n}(j_1, j_2) = s(j_1)^{-1/n} \cdot \delta(j_1, j_2). \qquad [33]$$

Finally, the continuity equation [25] yields $n_0 - 1$ conditions. Among them, $n_e - 1$ serve to express the external flow rates $\mathbf{j}_0^{(e)}$ as a function of the m "internal" flow rates $\mathbf{j}_{\text{int}}$; actually, these correspond to the continuity equations expressed at the $n_e - 1$ external vertices of Γ_0. The other $n_0 - 1 - (n_e - 1)$ conditions provide the relations necessary to determine the $n_0 - 1 - (n_e - 1)$ unknown components of N which remained to be calculated after the application of [31].

Such a distinction was unnecessary in VI where all the edges of the local graph had a non-zero conductance $s(j)$.

An example will be given in section 4.

A significant auxiliary aspect of the technique is its immediate extension to any nonlinear flow problem for which the pressure drop-flux relationship along an edge can be inverted. More precisely, in lieu of [19], one could obviously cope with the more general functional relationship:

$$\Delta p = -f[J(j)]. \tag{34}$$

In these more general circumstances, all of the formulae following [19] would continue to apply, provided that where appropriate, $M^{-1}(\ )^{1/n}$ was replaced by f^{-1}, symbolically representing the inverse of the function f.

4. NONLINEAR FLUIDS IN FRACTALS

The theoretical developments in this section follow very closely the ones for Stokes flow which were developed in II. First, some basic relations are recalled; then, the iteration formula, which relates two successive fractals, is derived. Finally, a particular attention is devoted to the examples; the fractal structure is the classical Sierpinski gasket; various power laws are studied.

4.1. *Basic relations*

Let us denote by $\mathbf{J}_N^{(e)}$ the vector defined on the subspace $V\Gamma_N^{(e)}$ of the external vertices of Γ_N; $\mathbf{J}_N^{(e)}$ represents the n_e flow rates going out of Γ_N. Similarly, the n_e pressures at these vertices are represented by the vector $\mathbf{P}_N^{(e)}$. Paralleling [7] and [11], the following relations hold for these two vectors:

$$\mathbf{J}_N^{(e)} = \mathbf{A}_N(\mathbf{P}_N^{(e)}), \tag{35a}$$

or equivalently,

$$\mathbf{j}_N^{(e)} = \mathbf{a}_N(\mathbf{p}_N^{(e)}). \tag{35b}$$

One of the major purposes of the present paper is to calculate in a general and convenient form these functions $\mathbf{A}_N$ and $\mathbf{a}_N$ as functions of the structure of the basic graph Γ_0 represented by its transfer function $\mathbf{A}_0$ or $\mathbf{a}_0$, as functions of the family of transformations $S_\alpha(\alpha = 1, \ldots, M)$ and as functions of the interconnections between the graphs $S_\alpha\Gamma_0(\delta = 1, \ldots, M)$ represented by the matrix ζ. Usually, iteration formulae are first obtained;

$$\mathbf{A}_N = \mathbf{F}_{s,\zeta}(\mathbf{A}_{N-1}) \tag{36a}$$

$$\mathbf{a}_N = \mathbf{f}_{s,\zeta}(\mathbf{a}_{N-1}), \tag{36b}$$

from which $\mathbf{A}_N$ and $\mathbf{a}_N$ are readily deduced as functions of $\mathbf{A}_0$ and $\mathbf{a}_0$:

$$\mathbf{A}_N = \overbrace{\mathbf{F}(\ldots\mathbf{F}(\mathbf{A}_0)\ldots)}^{N} \tag{37a}$$

$$\mathbf{a}_N = \overbrace{\mathbf{f}(\ldots\mathbf{f}(\mathbf{a}_0)\ldots)}^{N} \tag{37b}$$

The assemblage of graphs Γ_{N-1} which compose the graph Γ_N is now algebraically described. Γ'_N is defined (cf. [3]) as the juxtaposition without any interconnections of the M subgraphs $S_\alpha\Gamma_0$. An external flow rate vector $\mathbf{J}'^{(e)}_N$ can be defined on the space $V\Gamma'^{(e)}_N$ of all the external vertices of the graph Γ'_N; of course, $V\Gamma'^{(e)}_N$ is simply the union of all the external vertices of the transformed subgraphs $S_\alpha\Gamma_{N-1}$ ($\alpha = 1, \ldots, M$); hence,

$$V\Gamma'^{(e)}_N = \bigcup_{\alpha=1}^{M} S_\alpha(V\Gamma^{(e)}_{N-1}) \qquad [38]$$

$\mathbf{J}'^{(e)}_N$ is thus given by:

$$\mathbf{J}'^{(e)}_N = \begin{pmatrix} S_1\mathbf{J}^{(e)}_{N-1} \\ \vdots \\ S_\alpha\mathbf{J}^{(e)}_{N-1} \\ \vdots \\ S_M\mathbf{J}^{(e)}_{N-1} \end{pmatrix}, \qquad [39a]$$

and similarly,

$$\mathbf{P}'^{(e)}_N = \begin{pmatrix} S_1\mathbf{P}^{(e)}_{N-1} \\ \vdots \\ S_\alpha\mathbf{P}^{(e)}_{N-1} \\ \vdots \\ S_M\mathbf{P}^{(e)}_{N-1} \end{pmatrix}. \qquad [40]$$

On the graphs $\Gamma'_{N'}$ which is generally disconnected, the flow rate vector $\mathbf{J}'^{(e)}_N$ may be expressed as:

$$\mathbf{J}'^{(e)}_N = \begin{pmatrix} S_1[\mathbf{A}_{N-1}(S_1\mathbf{P}^{(e)}_{N-1})] \\ \vdots \\ S_\alpha[\mathbf{A}_{N-1}(S_\alpha\mathbf{P}^{(e)}_{N-1})] \\ \vdots \\ S_M[\mathbf{A}_{N-1}(S_\alpha\mathbf{P}^{(e)}_{N-1})] \end{pmatrix}. \qquad [41]$$

This expression may be condensed with the use of the vectorial function ζ,

$$\mathbf{J}'^{(e)}_N = \zeta(\mathbf{P}'^{(e)}_N). \qquad [42]$$

A word of comment is necessary here. ζ has $M \times n_e$ components. Its n_e components $(-1)\, n_e + 1$ to αn_e are equal to $S_\alpha[A_{N-1}\,(S_\alpha P^{(e)}_{N-1})]$.

Equation [42] may be rearranged as follows. Let us renumber the external flow rates and

$$\mathbf{P}_N'^{(e)i} = \mathbf{L}\dagger \cdot \mathbf{P}'^{i}. \qquad [49b]$$

With this formulation, [46] is automatically satisfied as a consequence of [48].

4.2. *Derivation of the transfer function*

As a pleasant consequence of these lengthy preliminaries, the formal solution of the problem is now at hand. Equation [44] may be split into two parts:

$$\mathbf{J}_N'^{(e)i} = \zeta_1(\mathbf{P}'^{(e)i}, \mathbf{P}_N^{(e)}), \qquad [50a]$$

$$\mathbf{J}_N^{(e)} = \zeta_2(\mathbf{P}'^{(e)i}, \mathbf{P}_N^{(e)}). \qquad [50b]$$

The solution is obtained as follows. The $(M-1)n_e$ relations [50a] furnish $(M-1)n_e/2$ relations, from which the components of the vector $\mathbf{P}'^{i}$ can be deduced. Introduction of these values into [50b] yields the desired relations between $\mathbf{J}_N^{(e)}$ and $\mathbf{P}_N^{(e)}$.

Let us now detail and discuss this solution. When [50a] is multiplied at its left by $\mathbf{L}$, its left-hand side vanishes as a consequence of [46a]. Introduction of [49b] yields:

$$\mathbf{L} \cdot \zeta_1(\mathbf{L}\dagger \cdot \mathbf{P}'^{i}, \mathbf{P}_N^{(e)}) = 0. \qquad [51]$$

This vectorial equality is equivalent to $(M-1)n_e/2$ relations between the $(M-1)n_e/2$ independent components of $\mathbf{P}'^{i}$ and the components of $\mathbf{P}_N^{(e)}$.

$\mathbf{P}'^{i}$ can be actually calculated as a function of $\mathbf{P}_N^{(e)}$ from [51]. A formal proof of this property would run parallel to the proof given in II for linear flows; in other words, it is based on the "invertibility" of $\mathbf{a}_0$ (cf. [11] and [12]). Moreover, the restrictions which accompany [12] imply that in general several possible solutions can be obtained in principle.

Hence, let us choose of these possible solutions and write

$$\mathbf{P}'^{i} = \phi_N(\mathbf{P}_N^{(e)}). \qquad [52]$$

Introduction of this equality and of [49b] into [50b] yields the final result,

$$\mathbf{J}_N^{(e)} = \zeta_2[\mathbf{L}\dagger \cdot \phi_N(\mathbf{P}_N^{(e)}), \mathbf{P}_N^{(e)}], \qquad [53]$$

which is the desired relation between $\mathbf{J}_N^{(e)}$ and $\mathbf{P}_N^{(e)}$. According to the definition [35a] of $\mathbf{A}_N$, we obtain:

$$\mathbf{A}_N = \zeta_2[\mathbf{L}\dagger \cdot \phi_N(\cdot), \cdot]. \qquad [54]$$

Of course, $\mathbf{A}_N$ is a function of $\mathbf{A}_{N-1}$ through ζ_2 and ϕ_N. In general, it is difficult to give an explicit dependence of ϕ_N upon $\mathbf{A}_{N-1}$ and thus to obtain an expression such as [36a]. Hence, a general iteration formula such as [36] is not explicitly given here.

4.3. *Power law fluids on a Sierpinski gasket*

Let us illustrate the general formulae, and their use, by the standard example of a Sierpinski gasket (cf. figure 3). Again fractal relations are shown to be obtained in the limit of large values of N; they will be systematically calculated as a function of the fluid index n.

The basic graph Γ_0 is a triangle (figure 3a); the associated graph γ_0 (figure 3b), on which the transfer function is calculated, possesses three vertices and $\overline{m} = 5$ edges. In order to avoid any possible confusion, the edges of γ_0 are denoted by a, b, c, d, e. A possible choice for the

pressures as:

$$\mathbf{J}_N'^{(e)} = \begin{pmatrix} \mathbf{J}_N'^{(e)i} \\ \mathbf{J}_N^{(e)} \end{pmatrix}, \tag{43a}$$

$$\mathbf{P}_N'^{(e)} = \begin{pmatrix} \mathbf{P}_N'^{(e)i} \\ \mathbf{P}_N^{(e)} \end{pmatrix}. \tag{43b}$$

The first $(M-1)n_e$ flow rates or pressures correspond to the $(M-1)n_e$ external vertices $v_i'^{(e)i}$ or Γ_N' which become internal vertices of Γ_N. The last n_e flow rates or pressures are thus relative to the n_e vertices $v_{N,i}^{(e)}$ of Γ_N' which become the final external vertices of Γ_N.

Hence, the functions ζ in [42] can be subdivided into two blocks as,

$$\begin{pmatrix} \mathbf{J}_N'^{(e)i} \\ \mathbf{J}_N^{(e)} \end{pmatrix} = \begin{pmatrix} \zeta_1(\mathbf{P}_N'^{(e)i}, \mathbf{P}_N^{(e)}) \\ \zeta_2(\mathbf{P}_N'^{(e)i}, \mathbf{P}_N^{(e)}) \end{pmatrix}. \tag{44}$$

When the M subgraphs $S_\alpha\Gamma_{N-1}(\alpha = 1, \ldots, M)$ are connected according to the law symbolized by ζ (cf. [16]), it implies that the flow rates are equal and opposite (recall the algebraic convention) and that the pressures are equal at the vertices which are connected one to the other one. This may be expressed as:

$$\zeta \cdot \mathbf{J}_N'^{(e)i} = -\mathbf{J}_N'^{(e)i} \tag{45a}$$

$$\zeta \cdot \mathbf{P}_N'^{(e)i} = \mathbf{P}_N'^{(e)i} \tag{45b}$$

which provide $2(M-1)n_e$ relations between flow rates and pressures. It was shown in II that the relations [45] can be replaced by the nonredundant set of $(M-1)n_e$ relations:

$$\mathbf{L} \cdot \mathbf{J}_N'^{(e)i} = 0 \tag{46a}$$

$$\mathcal{L} \cdot \mathbf{P}_N'^{(e)i} = 0 \tag{46b}$$

where $\mathbf{L}$ and $\mathcal{L}$ are $(M-1)n_e/2 \times (M-1)n_e$ matrices obtained as follows. First, the matrix ζ^+ is defined as:

$$\begin{aligned} \zeta_{ij}^{\ +} &= \zeta_{ij}, \qquad i < j \\ &= 0, \qquad i > j. \end{aligned} \tag{47}$$

Then $+1$ (for $\mathbf{L}$) or -1 (for $\mathcal{L}$) is added to the first diagonal of ζ^+ each time that there is already 1 in the corresponding line. Then the matrices $\mathbf{L}$ and $\mathcal{L}$ are obtained by deleting the lines composed of 0 only. These matrices enjoy the elementary properties:

$$\mathbf{L} \cdot \mathcal{L}\dagger = 0, \mathbf{L} \cdot \mathbf{L}\dagger = 2\mathbf{I}, \mathcal{L} \cdot \mathcal{L}\dagger = 2\mathbf{I}. \tag{48}$$

Finally, it proved convenient to introduce new and nonredundant flow rate and pressure vectors $\mathbf{J}'^i$ and $\mathbf{P}'^i$, each of them possessing $(M-1)n_e/2$ independent components, and such that:

$$\mathbf{J}_N'^{(e)i} = \mathcal{L}\dagger \cdot \mathbf{J}'^i, \tag{49a}$$

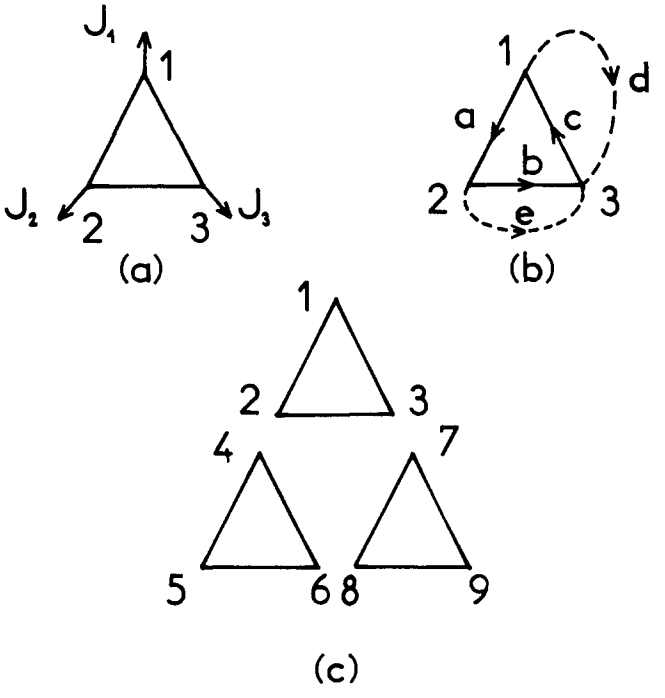

Figure 3. The Sierpinski gasket. (a) The basic graph Γ_0. (b) The graph γ_0; the edges are numbered by the letters $a, b \ldots e$. (c) Construction of the gasket; the first generation Γ_1.

cycle and cocycles matrices **C** and **K** is:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{55}$$

It is then straightforward to show that:

$$\mathbf{C} \cdot (\mathbf{C}\dagger \cdot \mathbf{C})^{-1} \cdot \mathbf{C}' = \frac{1}{8} \begin{pmatrix} 4 & 2 & 2 & -2 & 2 \\ 2 & 5 & 1 & -1 & -3 \\ 2 & 1 & 5 & 3 & 1 \\ -2 & -1 & 3 & 5 & -1 \\ 2 & -3 & 1 & -1 & 5 \end{pmatrix}. \tag{56}$$

The three edges, a, b and c, are supposed to have the same resistance S which is assumed equal to 1. Note that all quantities in this paragraph are dimensionless, pressures, flow rates. Hence **M** (cf. [21]) is deduced to be:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{57}$$

The pressure generator vector **g** is given by (cf. [22]):

$$\mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ p_1 \\ p_2 \end{pmatrix}, \qquad [58]$$

when the origin of the pressures is located at vertex 3.

The arbitrary vector **N** (cf. [30]) has two components N_1 and N_2. An easy utilization of the previous formulae enables us to write [29a] as:

$$\mathbf{M} \cdot \bar{j}_0^n = -\frac{1}{8} \cdot \begin{pmatrix} -2p_1 + 2p_2 \\ -p_1 - 3p_2 \\ 3p_1 + p_2 \\ 5p_1 - p_2 \\ -p_1 + 5p_2 \end{pmatrix} - \begin{pmatrix} N_1 + N_2 \\ -N_2 \\ -N_1 \\ N_1 \\ -N_2 \end{pmatrix}. \qquad [59]$$

The continuity equation [25] yields two conditions. Note that we are here in a particular situation since the number of vertices of Γ_0 is equal to the number $n_e = 3$ of external vertices of Γ_0. Hence, we obtain the external flow rates as functions of the internal ones (cf. the comments after [33]),

$$\begin{aligned} \bar{j}_{0,d} &= \bar{j}_{0,c} - \bar{j}_{0,a} \\ \bar{j}_{0,e} &= \bar{j}_{0,a} - \bar{j}_{0,b}. \end{aligned} \qquad [60]$$

Finally, the last two equations of [59] yield two relations between **N** and **g** (cf. [31]),

$$8N_1 = p_2 - 5p_1, \qquad 8N_2 = -p_1 + 5p_2. \qquad [61]$$

These values can be inserted into the three first equations [59]. The flows on the internal edges of Γ_0 are deduced to be:

$$\begin{pmatrix} \bar{j}_{0,a} \\ \bar{j}_{0,b} \\ \bar{j}_{0,c} \end{pmatrix} = \begin{pmatrix} p_1 - p_2 \\ p_2 \\ -p_1 \end{pmatrix}^{1/n}. \qquad [62]$$

Of course, in such a simple situation, these relations could have been written down directly, but this is not always the case. Going back to the basic graph Γ_0, the outgoing flow rates J_1, J_2 and J_3 may be written as:

$$\begin{aligned} J_1 &= -(P_1 - P_3)^{1/n} - (P_1 - P_2)^{1/n}, \\ J_2 &= (P_1 - P_2)^{1/n} - (P_2 - P_3)^{1/n}, \\ J_3 &= (P_2 - P_3)^{1/n} + (P_1 - P_3)^{1/n}. \end{aligned} \qquad [63]$$

These three equations constitute the explicit expression of the transfer function $\mathbf{A}_0$ (cf. [7]).

Let us now calculate the transfer function $\mathbf{A}_1$ of the graph Γ_1 which is illustrated in figure

3c. The external vertices of Γ_1 are ordered as 1, 5, 9 and the internal vertices as 2, 3, 4, 6, 7 and 8. The matrices ζ and $\mathbf{L}$ may be expressed as (cf. II for more details):

$$\zeta = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{L} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{64}$$

The nonlinear vectorial functions ζ_1 and ζ_2 (cf. [44]) and obtained by writing three times [63] for the three basic graphs Γ_0 which compose Γ_1. Application of [51] yields the nonlinear system:

$$\begin{aligned} (P_1 - P_2)^{1/n} + (P_3 - P_2)^{1/n} + (P_6 - P_2)^{1/n} + (P_5 - P_2)^{1/n} &= 0, \\ (P_1 - P_3)^{1/n} + (P_2 - P_3)^{1/n} + (P_6 - P_3)^{1/n} + (P_9 - P_3)^{1/n} &= 0, \\ (P_2 - P_6)^{1/n} + (P_5 - P_6)^{1/n} + (P_3 - P_6)^{1/n} + (P_9 - P_6)^{1/n} &= 0, \end{aligned} \tag{65}$$

from which P_2, P_3 and P_6 can be calculated as functions of P_1, P_5 and P_9.

Once these intermediate pressures are known, $\mathbf{A}_1$ may be obtained by a straightforward application of [53].

So far, the general method was faithfully followed for illustration purposes and it is gratifying to check that our general formalism works properly. However, power law fluids possess a general feature which will greatly shorten the numerical computations; namely, the transfer function $\mathbf{A}_0$ is a homogeneous function. When all the pressures are multiplied by a factor c, the flow rates are multiplied by $c^{1/n}$. It is easily shown that the transfer functions $\mathbf{A}_1, \ldots, \mathbf{A}_N$ are also homogeneous with the same exponent n. Hence, it is sufficient to calculate $\mathbf{A}_0 \ldots \mathbf{A}_N$ on the unit circle in the following way.

On Γ_0, let us assume that the origin of pressures is P_1. Since the vertices 2 and 3 play a symmetric role in Γ_0, [63] may be written with a unique function f_0 of the two arguments p_2 and p_3:

$$\begin{aligned} J_1 &= -f_0(p_2, p_3) - f_0(p_3, p_2), \\ J_2 &= f_0(p_2, p_3), \\ J_3 &= f_0(p_3, p_2). \end{aligned} \tag{66}$$

f_0 is easily derived from a comparison between [63] and [66].

Since f_0 is homogeneous, it suffices to know it on the unit circle, i.e. for p_2 and p_3 such that:

$$(p_2)^2 + (p_3)^2 = 1. \tag{67}$$

In general, p_2 and p_3 may be expressed as:

$$p_2 = \lambda \cos\theta, \qquad p_3 = \lambda \sin\theta \tag{68}$$

whence

$$\begin{aligned} f_0(p_2, p_3) &= \lambda^{1/n} \cdot f_0(\cos\theta, \sin\theta) \\ &= \lambda^{1/n} \cdot f_0(\theta). \end{aligned} \tag{69}$$

In order to keep simple notations, f_0 denotes two different functions in [69].

This is easily generalized for any generation N. When f_N is known, f_{N+1} is calculated as follows. p_2, p_3 and p_6 are calculated as functions of p_5 and p_9 from the system deduced from [65]:

$$\begin{aligned} f_N(p_2, p_3) + f_N(p_2 - p_5, p_6 - p_5) &= 0, \\ f_N(p_3, p_2) + f_N(p_3 - p_9, p_6 - p_9) &= 0, \\ f_N(p_6 - p_5, p_2 - p_5) + f_N(p_6 - p_9, p_3 - p_9) &= 0. \end{aligned} \qquad [70]$$

p_5 and p_9 are assumed to belong to the unit circle and to correspond to an angle θ. The function $f_{N+1}(\theta)$ is then expressed as:

$$\begin{aligned} J_5 &= f_{N+1}(\theta) \\ &= -f_N(p_4 - p_5, p_6 - p_5) - f_N(p_6 - p_5, p_4 - p_5). \end{aligned} \qquad [71]$$

These calculations are best done on a computer; the numerical calculation can proceed as follows. f_N is assumed to be known for a certain number of values of the parameter θ. An IMSL subroutine is used to interpolate when necessary between these values with a cubic spline. Then for a given value of θ, i.e. of p_5 and p_9, the nonlinear system [70] is solved by iteration. $f_{N+1}(\theta)$ is then obtained by applications of [71].

Finally, let us present and briefly discuss the results. Some details are first exposed for the particular case of a fluid index equal to 0.5 $f_0(\theta)$ is represented in figure 4. The positive branch of $f_N(\theta)$ is shown in figure 5 for various values of N. As expected from our previous results on linear fluids (cf. II), the transfer function $f_N(\theta)$ obeys very rapidly the fractal relation:

$$f_N(\theta) = K(n) f_{N-1}(\theta) \qquad \text{for } N \text{ large}, \qquad [72]$$

where $K(n)$ is a constant which only depends upon the fluid index n. Hence, in the semilogarithmic representation of figure 5, the various functions are deduced one from the other one by translations.

The function $K(n)$ was then systematically determined. Results are displayed in figure 6. For $n = 1$, i.e. for a Newtonian fluid, the classical constant $\frac{3}{5}$ is again obtained (cf. II) and this serves as a useful check. When n is small, the fluid is shear-thinning, and $K(n)$ tends toward 0. When n is large, the fluid is shear-thickening and $K(n)$ tends toward 1.

So far we have not been able to derive the constant $K(n)$ on simple grounds, mainly because the star-triangle transformation is not valid anymore here.

5. BINGHAM FLUIDS

This section is devoted to the study of Bingham fluids on fractals; it is interesting to determine the minimal pressure difference required to initiate flow between two external vertices of the fractals. First, a general method is presented in order to determine it in a systematic way; some basic features of Bingham fluids are recalled. Finally, the Sierpinski gasket is briefly analysed.

Oldroyd's commonly accepted multidimensional generalization of Bingham's original one-dimension equation is (Bird *et al.* 1960; Fredrickson 1960):

$$\begin{aligned} \mathbf{S} &= 0 \qquad \text{for } \frac{1}{2} \cdot \mathbf{T}:\mathbf{T} < T_0^2 \\ \mathbf{T} &= 2\left[\mu_\infty + \frac{T_0}{(2\mathbf{S}:\mathbf{S})^{1/2}}\right] \cdot \mathbf{S} \qquad \text{for } \frac{1}{2} \cdot \mathbf{T}:\mathbf{T} > T_0^2, \end{aligned} \qquad [73]$$

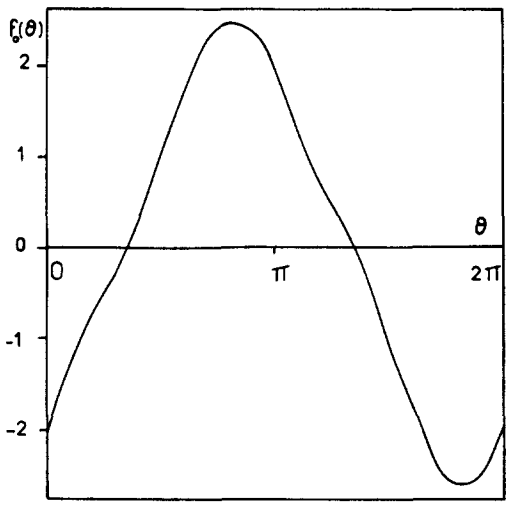

Figure 4. The function $f_0(\theta)$ for the basic graph Γ_0. Value of n: 0.5.

where **S** is the rate-of-strain tensor, **T** the stress tensor, T_0 the yield stress, and μ_∞ a phenomenological constant representing the Newtonian viscosity prevailing at infinite strain rate.

Equation [73] will not be explicitly used here, except with regard to one-dimensional applications involving flow in circular capillaries. In this case, the flow initiation condition is (Fredrickson 1964),

$$-\frac{2\,T_0}{r\,|\nabla p\,|} < 1, \qquad [74]$$

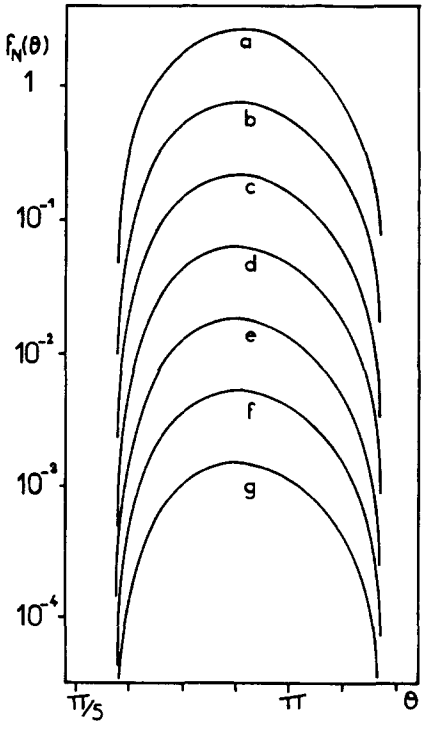

Figure 5. The positive branch of $f_N(\theta)$ for the fractal graphs Γ_N. Value of n: 0.5. Values of N are: 0(a), 1(b), 2(c), 3(d), 4(e), 5(f), 6(g).

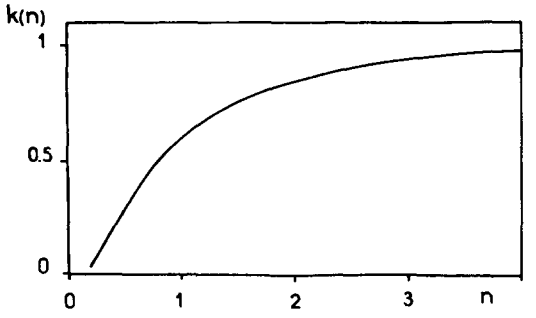

Figure 6. The fractal constant $K(n)$ for a Sierpinski gasket as a function of the fluid index n.

with the scalar $|\nabla p|$ the prescribed pressure gradient in the capillary of radius r. Similar criteria, but with different numerical factors, prevail for noncircular capillaries.

Hence, according to [74], a positive magnitude can be assigned to each of the basic graph Γ_0. This number represents the minimum value of the pressure differential along an edge j that will still permit the fluid to flow locally. For instance, for a circular capillary of length $L(j)$ and radius $r(j)$ this number would possess the magnitude:

$$T(j) = \frac{2\,T_0}{r(j)} \cdot L(j). \qquad [75]$$

Again, this amounts to assume that the phenomena at the capillary junctions can be neglected. This is a safe assumption in the same conditions as for the power-law fluids.

Let us consider two arbitrary external vertices i_0 and i'_0 of Γ_0. The threshold $T_0(i_0, i'_0)$ can be calculated; it corresponds to the shortest route between i_0 and i'_0 when each edge is given the "length" $T(j)$. It may be expressed as:

$$T_0(i_0, i'_0) = \min\left\{\sum_{j\in w} T(j);\ W\text{: family of the walks between } i_0 \text{ and } i'_0\right\}. \qquad [76]$$

Obviously, flow is initiated between i_0 and i'_0 when

$$|p(i_0) - p(i'_0)| \geq T_0(i_0, i'_0). \qquad [77]$$

Moreover, it occurs along the walk(s) which correspond(s) to the minimum value of the summation in [76].

When the basic graphs Γ_0 are assembled in order to form the fractal Γ_1, the threshold $T_1(i_1, i'_1)$ corresponding to the shortest route between the two external vertices i_1 and i'_1 of Γ_1 may be expressed as:

$$T_1(i_1, i'_1) = \min\left\{\sum_{W} T_0(i_{0\alpha}, i_{0\beta});\ W\text{: family of the walks between } i_1 \text{ and } i'_1\right\}. \qquad [78]$$

where $i_{0\alpha}$ and $i_{0\beta}$ are external vertices of the basic graph Γ_0.

This formula may be easily generalized to the fractal Γ_N.

$$T_N(i_N, i'_N) = \min\left\{\sum_{W} T_{N-1}(i_{N-1,\alpha}, i_{N-1,\beta});\ W\text{: family of the walks between } i_N \text{ and } i'_N\right\}. \qquad [79]$$

Hence, the search on the shortest route on the fractal is facilitated by the fact that the route has been minimized on each subgraph Γ_{N-1}, which forms Γ_N.

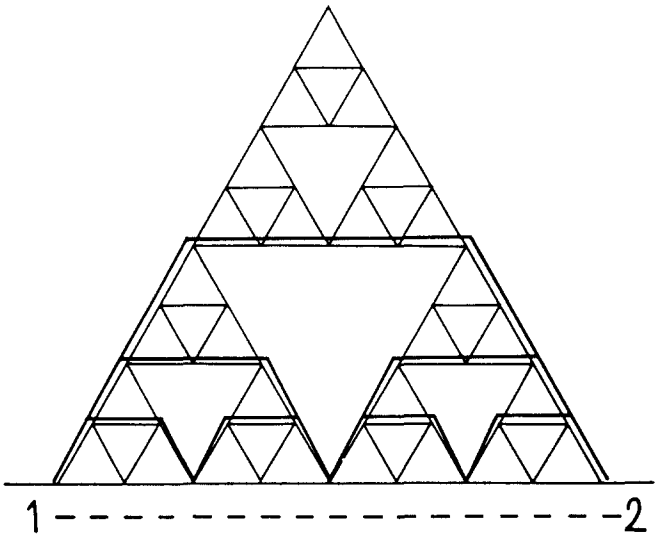

Figure 7. Bingham fluid in a Sierpinski gasket. When a pressure drop is imposed between 1 and 2, the flow starts flowing along the side (1, 2) as indicated by the broken line. When $|P_1 - P_2|$ is further increased, new flow routes are simultaneously created, as indicated by the solid thick lines. The description of further steps would require the detailed knowledge of fluid properties.

Note that a very different problem occurs when fluid flows between three or more vertices; in this case, there may be interaction between the various flow paths. This possible interaction becomes more and more complicated when the fractal gets more and more involved.

The Sierpinski gasket is considered in order to illustrate the previous developments. It is shown in figure 7. Obviously, when all the edges of the basic graph Γ_0 are assigned the same threshold, the shortest route between two vertices of the gasket is the side of the triangle.

One could easily go a little bit further and show how the flow progressively invades the inner structures of the graph. The second step is illustrated in figure 7. The determination of the further steps requires the knowledge of μ_∞ and T_0 (cf. [73]).

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